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Some Fixed Point Results on Multiplicative Metric Spaces

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Abstrak. Artikel ini diawali dengan pembahasan beberapa teori terkaitruang ruang metrik multiplikatif untuk mendukung hasil utama. Tujuan utama dari artikel ini adalah untuk membuktikan beberapa teorema titik tetap yang memenuhi beberapa perumuman dari pemetaan kontraksi dalam kaitannya dengan ruang metrik multiplikatif. Lebih lanjut, beberapa contoh diberikan untuk memperkuat hasil yang diperoleh.

Kata kunci: titik tetap, ruang metrik multiplikatif

Abstract. *In this paper, we first discussed some results about multiplicative metric spaces to support the main results. The aim of this paper is to present some fixed point results that satisfied some generalized of contraction mapping related to multiplicative metric spaces. Furthermore, some examples are given to support results.*

Keywords*:fixed point, multiplicative metric spaces*

I. INTRODUCTION

Multiplicative metric space is new kind of metric spaces that was introduced by Bashirov et. al [1]. Afterwards, Ozavsar and Cevikel [2] prove some properties of multiplicative metric space and discussed some fixed points results in multiplicative metric spaces. On the other hands, some research also makes some remark about multiplicative metric space such as Shukla [3] and Agarwal et. al [4].

Some researchers have been made improve and generalized about contraction mapping in prove fixed point results. .Dosenaovic and Radenovic [5] prove the existence of fixed point of contractions rational type in multiplicative metric spaces. Also, Abdou [6] prove fixed point theorems for generalized contraction mappings in multiplicative metric spaces. Other result about fixed point theory in multiplicative metric space can be seen in [7], [8], [9], [10], and [11]

In this article, we prove some fixed points results in multiplicative metric space. We make some generalizations from previous results. Also, some examples be given to illustrate some main results.

II. PRELIMINARIES

In this section, we recall some definitions and theorems related to multiplicative metric spaces.

Definition 1.[1] Let X be nonempty set. A mapping $d^*: X\times X \to \mathbf{R}$ is called multiplicative metric on X if satisfy the following conditions for all $x,y,z\in X$

(1) $d^*(x,y)\geq 1$

(2) $d^*(x,y)=1$ if and only if $x=y$

(3) $d^*(x,y)=d^*(y,x)$

(4) $d^*(x,z)\leq d^*(x,y)d^*(y,z)$

A pair (X,d^*) is called multiplicative metric space.

In multiplicative metric spaces (X,d^*), we define multiplicative open ball of $x\in X$ with radius $\varepsilon>1$ as

$$B_\varepsilon(x)=\{y\in X \,|\, d^*(x,y)<\varepsilon\}$$

and multiplicative close ball as

$$\bar{B}_\varepsilon(x)=\{y\in X \,|\, d^*(x,y)\leq\varepsilon\}$$

Example 1. [12]

(i) Let (X,d) be metric spaces. For real numbers $a>1$ we defined $d^*(x,y)=a^{d(x,y)}$ for $x,y\in X$. It is easy to check that $d^*(x,y)$ is multiplicative metric on X.

(ii) Discrete multiplicative metric spaces is defined as

$$d^*(x,y)=a^{d(x,y)}=\begin{cases}1 & \text{if } x=y\\ a & \text{if } x\neq y\end{cases}$$

for $x,y\in X$ where d is standard metric on X

Theorem 1. [4]
Let (X,d^*) be multiplicative metric spaces, then the mapping $d: X\times X \to \mathbf{R}^+$ with $d(x,y)=\ln\left(d^*(x,y)\right)$ forms a metric on X.

From above theorem, we know that all fixed points theorem in multiplicative metric space implies fixed point theorem related to metric $d(x,y)=\ln\left(d^*(x,y)\right)$

Definition 2. [2]
Let (X,d^*) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x\in X$. A sequence $\{x_n\}$ is said to multiplicative convergent to x, written $x_n \to_* x$, if every multiplicative open ball $B_\varepsilon(x)$, there exist $K\in\mathbf{N}$ such that for $n\geq K$ implies $x_n \in B_\varepsilon(x)$.
Equivalently, sequence $\{x_n\}$ convergent to x, if for every $\varepsilon>1$ there exist $K\in\mathbf{N}$ such that for $n\geq K$ implies $d^*(x_n,x)<\varepsilon$.

Lemma 2. [2]
Let (X,d^*) be multiplicative metric spaces, $\{x_n\}$ be a sequence in X and $x\in X$. Sequence $\{x_n\}$ multiplicative convergent to x if and only if $d^*(x_n,x)\to_* 1$ as $n\to\infty$

Definition 3. [2]
Let (X,d^*) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. A sequence $\{x_n\}$ is said to multiplicative Cauchy sequence if for all $\varepsilon>1$ there exist $K\in\mathbf{N}$ such that for all $m,n\geq K$ implies $d^*(x_n,x_m)<\varepsilon$.

Definition 4. [2]
A multiplicative metric space (X,d^*) is said to complete if every multiplicative Cauchy sequence in X convergence in X.

Theorem 3. [2]
Let (X,d^*) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is multiplicative convergent, then $\{x_n\}$ is multiplicative Cauchy sequence.

Theorem 4. [2]
Let (X,d^*) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. A sequence $\{x_n\}$ is multiplicative Cauchy
If and only if $d^*(x_n,x_m)\to_* 1$ for $m,n\to\infty$.

The following definition discuss about multiplicative contraction mapping and explain about some results of fixed point theorems.

Definition 5. [2]
Let (X,d^*) be a multiplicative metric space. A mapping $T: X\times X$ is called multiplicative contraction mapping if there exist real numbers $0<\alpha<1$ such that for all $x,y\in X$

$$d^*(Tx,Ty)\leq d^*(x,y)^\alpha$$

Theorem 5. [2]
Let (X,d^*) be a multiplicative metric space and $T: X\times X$ be a multiplicative contraction mapping. If (X,d^*) is complete, then T has unique fixed point.

Theorem 6. [12]
Let (X,d^*) be a multiplicative metric space and a,b,c be real numbers with $a\in(0,1)$ dan $b,c\in\left(0,\frac{1}{2}\right)$. If mapping $T: X\times X$ satisfied at least one of the following conditions

1. $d^*(Tx,Ty)\leq\left(d(x,y)\right)^a$
2. $d^*(Tx,Ty)\leq\left(d^*(x,Tx).d^*(y,Ty)\right)^b$
3. $d^*(Tx,Ty)\leq\left(d^*(x,Ty).d^*(y,Tx)\right)^c$

for all $x,y\in X$, then T has a unique fixed point.

III. RESULTS AND DISCUSSION

In the following, we provide some fixed points results on the multiplicative metric spaces.

Theorem 7.

Let (X,d^*) be a complete multiplicative metric space. If $T: X\to X$ be a mapping that satisfied

$$d^*(Tx,Ty)\leq d^*(x,Tx)^a\cdot d^*(y,Ty)^b\cdot d^*(x,y)^c \quad (1)$$

with a,b,c are nonnegative real numbers and satisfy $a+b+c<1$, then T has unique fixed point.

Proof.
Let $x_0\in X$ is any point. For all $n\in\mathbf{N}$, we define a sequence

$$x_n=T^n x_0=T\left(T^{n-1}x_0\right)=Tx_{n-1}$$

We have

$$\begin{aligned} d^*(x_{n+1},x_n) &= d^*(Tx_n,Tx_{n-1}) \\ &\leq d^*(x_n,Tx_n)^a\, d^*(x_{n-1},Tx_{n-1})^b\, d^*(x_n,x_{n-1})^c \\ &\leq d^*(x_n,x_{n+1})^a\, d^*(x_{n-1},x_n)^b\, d^*(x_n,x_{n-1})^c \\ &\leq d^*(x_n,x_{n+1})^a\, d^*(x_{n-1},x_n)^{b+c} \end{aligned}$$

Therefore,

$$d^*(x_{n+1},x_n)\leq d^*(x_{n-1},x_n)^{\frac{b+c}{1-a}}$$

Let $\alpha=\dfrac{b+c}{1-a}$, we get

$$d^*(x_{n+1},x_n)\le d^*(x_1,x_0)^{\alpha^n}$$

For any $m,n\in\mathbf{N}$, with $m>n$

$$d^*(T^m x_0,T^n x_0)\le d^*(T^m x_0,T^{m-1}x_0)\cdot d^*(T^{m-1}x_0,T^{m-2}x_0)\ldots d^*(T^{n+1}x_0,T^n x_0)$$
$$\le d^*(x_1,x_0)^{\alpha^{(m-1)}+\alpha^{(m-2)}+\ldots+\alpha^n}$$
$$\le d^*(x_1,x_0)^{\frac{\alpha^n-\alpha^m}{1-\alpha}}$$

Since $a+b+c<1$, then $\alpha<1$, therefore

$$\lim_{m,n\to\infty} d^*(T^m x_0,T^n x_0)=1$$

So, sequence $\{x_n\}$ is multiplicative Cauchy sequence in X. Since X is complete multiplicative metric space, then there exist $x\in X$ such that $x_n\to_* x$.

We show that x is fixed point of T. Therefore

$$\begin{aligned}d^*(x,Tx)&\le d^*(x,x_n).d^*(x_n,Tx)\\&=d^*(x,x_n).d^*(Tx_{n-1},Tx)\\&\le d^*(x,x_n).d^*(x,Tx)^a.d^*(x_{n-1},Tx_{n-1})^b.\\&\quad d^*(x,x_{n-1})^c\end{aligned}$$

We get

$$d^*(x,Tx)\le d^*(x_n,x)^{\frac{1}{1-a}}.d^*(x_{n-1},x_n)^{\frac{b}{1-a}}.d^*(x,x_{n-1})^{\frac{c}{1-a}}$$

Since $x_n\to_* x$, from Lemma 2 we obtain $Tx=x$. It shows that $x\in X$ is a fixed point of T.

Now, we show that T has a unique fixed point. Assume that there are two fixed point of T. Let $x,y\in X$ are fixed point of T with $x\neq y$. So, we have

$$\begin{aligned}d^*(x,y)&=d^*(Tx,Ty)\\&\le d^*(x,Tx)^a.d^*(y,Ty)^b.d^*(x,y)^c\\&=d^*(x,x)^a.d^*(y,y)^b.d^*(x,y)^c\\&=d^*(x,y)^c\end{aligned}$$

This implies $d^*(x,y)=1$ or $x=y$. This show that T has unique fixed point. ■

Corollary 8.

Let (X,d^*) be a complete multiplicative metric space. If $T:X\to X$ be a mapping that satisfied

$$d^*(T^n x,T^n y)\le d^*(x,T^n x)^a\cdot d^*(y,T^n y)^b\cdot d^*(x,y)^c \quad (2)$$

for all $x,y\in X$ with a,b,c are nonnegative real numbers and satisfy $a+b+c<1$, then T has unique fixed point. ■

Proof.

Suppose that $Sx=T^n x$. So, by Theorem 7, there exist fixed point of S $z\in X$ such that $Sz=z$.

So, $T^n z=z$. Therefore,

$$\begin{aligned}d^*(Tz,z)&=d^*(T(T^n z),z)\\&=d^*(T^n(Tz),T^n z)\\&\le d^*(Tz,T^n(Tz))^a.d^*(z,T^n z)^b.d^*(Tz,z)^c\\&=d^*(Tz,T(T^n z))^a.d^*(z,z)^b.d^*(Tz,z)^c\\&=d^*(Tz,z)^c\end{aligned}$$

This implies $d^*(Tz,z)=1$ or $Tz=z$. This show that T has fixed point. We show that fixed point of T is unique. Let $z,y\in X$ are fixed point of T with $z\neq y$. So, we have

$$\begin{aligned}d^*(z,y)&=d^*(T^n z,T^n y)\\&\le d^*(z,T^n z)^a.d^*(y,T^n y)^b.d^*(z,y)^c\\&=d^*(z,z)^a.d^*(y,y)^b.d^*(z,y)^c\\&=d^*(z,y)^c\end{aligned}$$

This implies $d^*(z,y)=1$ or $z=y$. This show that T has unique fixed point. ■

The next result will discuss about fixed point of weakly multiplicative contraction mapping using control functions ϕ defined by Abbas et.al. [13].

Definition 6

Let (X,d^*) be a multiplicative metric space. A mapping T is called weakly multiplicative contraction mapping is there exist continuous non-decreasing function $\phi:[1,\infty)\to[1,\infty)$ with $\phi(t)=1$ if and only if $t=1$ such that

$$d^*(Tx,Ty)\le\frac{d^*(x,y)}{\phi(d^*(x,y))} \quad (3)$$

for all $x,y\in X$.

It is easy to check that if we choose $\phi(t)=t^{1-\alpha}$ for $0<\alpha<1$, we get $d^*(Tx,Ty)\le d^*(x,y)^\alpha$. So, every multiplicative contraction mapping is weakly multiplicative contraction mapping. The following theorem prove the existence of fixed point of multiplicative contraction mapping.

Theorem 9.

Let (X,d^*) be a complete multiplicative metric space and $T:X\times X$ be a weakly multiplicative contraction mapping. If (X,d^*) is complete, then T has unique fixed point.

Proof.

Consider x_0 be a point in X. We define a sequence $\{x_n\}$ with $x_n = Tx_{n-1} = T^n x_0$. We have

$$\begin{aligned} d^*(x_{n+2}, x_{n+1}) &= d^*(Tx_{n+1}, Tx_n) \\ &\leq \frac{d^*(x_{n+1}, x_n)}{\phi(d^*(x_{n+1}, x_n))} \\ &\leq d^*(x_{n+1}, x_n) \end{aligned}$$

Therefore, a sequence $\{\rho_n\} = \{d^*(x_{n+1}, x_n)\}$ is decreasing positive sequence and bounded below by 1. Consequently, the sequence $\{\rho_n\}$ converge. Let sequence $\{\rho_n\}$ converge to $\rho \geq 1$.

Since ϕ is nondecreasing, then

$$\phi(\rho_n) \geq \phi(\rho)$$

We get

$$\rho_{n+1} \leq \frac{\rho_n}{\phi(\rho_n)}$$

For $m \in \mathbf{N}$, we have

$$\rho_{n+m} \leq \frac{\rho_n}{\phi(\rho_n)^m}$$

Taking limit for $n \to \infty$, then

$$\rho \leq \frac{\rho}{\phi(\rho)^m}$$

Therefore, $\phi(\rho) = 1$ or $\rho = 1$. So, sequence $\{\rho_n\}$ converge multiplicative to 1 or $\lim_{n\to\infty} d^*(x_{n+1}, x_n) = 1$.

Next, we will prove that sequence $\{x_n\}$ is multiplicative Cauchy sequence. Claim that $\lim_{n,m\to\infty} d^*(x_n, x_m) = 1$. If $\{x_n\}$ is not multiplicative Cauchy sequence, then there exist $\varepsilon > 1$ and sequence $\{n_k\}, \{m_k\} \subseteq \mathbf{N}$ such that $d^*(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbf{N}$ with $n_k > m_k \geq k$. Therefore, we can assume that $d^*(x_{n_k}, x_{m_k-1}) < \varepsilon$. Since $\lim_{n\to\infty} d^*(x_{n+1}, x_n) = 1$, we have $\lim_{k\to\infty} d^*(x_{m_k-1}, x_{m_k}) = 1$ or subsequence $\{d^*(x_{m_k-1}, x_{m_k})\}$ multiplicative converge to 1. Consider

$$\varepsilon \leq d^*(x_{n_k}, x_{m_k}) \leq d^*(x_{n_k}, x_{m_k-1}).d^*(x_{m_k-1}, x_{m_k})$$

Implies that

$$\lim_{k\to\infty} d^*(x_{m_k}, x_{n_k}) = \varepsilon \qquad (4)$$

Since $d^*(x_{m_k}, x_{n_k}) \leq d^*(x_{m_k}, x_{m_k-1}).d^*(x_{m_k-1}, x_{n_k})$, then $\varepsilon \leq \lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k})$. In addition, from (4) and $d^*(x_{m_k-1}, x_{n_k}) \leq d^*(x_{m_k-1}, x_{m_k}).d^*(x_{m_k}, x_{n_k})$, then $\lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k}) \leq \varepsilon$. Consequently,

$$\lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k}) = \varepsilon \qquad (5)$$

From equation (5) $d^*(x_{m_k-1}, x_{n_k}) \leq d^*(x_{m_k-1}, x_{n_k+1}).d^*(x_{n_k+1}, x_{n_k})$, then $\varepsilon \leq \lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k+1})$. Also, from (5) and $d^*(x_{m_k-1}, x_{n_k+1}) \leq d^*(x_{m_k-1}, x_{n_k}).d^*(x_{n_k}, x_{n_k+1})$, then $\lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k+1}) \leq \varepsilon$. Consequently,

$$\lim_{k\to\infty} d^*(x_{m_k-1}, x_{n_k+1}) = \varepsilon \qquad (6)$$

From equation (4) $d^*(x_{m_k}, x_{n_k}) \leq d^*(x_{m_k}, x_{n_k+1}).d^*(x_{n_k+1}, x_{n_k})$, then $\varepsilon \leq \lim_{k\to\infty} d^*(x_{m_k}, x_{n_k+1})$. Also, from (4) and $d^*(x_{m_k}, x_{n_k+1}) \leq d^*(x_{m_k}, x_{n_k}).d^*(x_{n_k}, x_{n_k+1})$, then $\lim_{k\to\infty} d^*(x_{m_k}, x_{n_k+1}) \leq \varepsilon$. Consequently,

$$\lim_{k\to\infty} d^*(x_{m_k}, x_{n_k+1}) = \varepsilon \qquad (7)$$

From (3), we get

$$\begin{aligned} d^*(x_{n_k+1}, x_{m_k}) &= d^*(Tx_{n_k}, Tx_{m_k-1}) \\ &\leq \frac{d^*(x_{n_k}, x_{m_k-1})}{\phi(d^*(x_{n_k}, x_{m_k-1}))} \end{aligned}$$

Taking limit of both sides as $k \to \infty$ we have $\varepsilon \leq \frac{\varepsilon}{\phi(\varepsilon)}$ or $\phi(\varepsilon) = 1$ implies that $\varepsilon = 1$. Contradiction with $\varepsilon > 1$. Hence, our claim that $\lim_{n,m\to\infty} d^*(x_n, x_m) = 1$ is true. So, sequence $\{x_n\}$ is multiplicative Cauchy sequence.

Since (X, d^*) is complete metric spaces, sequence $\{x_n\}$ is multiplicative converge in X. Let $\{x_n\}$ converge to $x \in X$. Hence, there exist $N \in \mathbf{N}$ such that for all $n \geq N$ implies $d^*(x_n, x) < \varepsilon$. For all $n \geq N$, we get

$$\begin{aligned} d^*(Tx, x) &\leq d^*(Tx, x_{n+1}).d^*(x_{n+1}, x) \\ &= d^*(Tx, Tx_n).d^*(x_{n+1}, x) \\ &< \frac{d^*(x, x_n)}{\phi(d^*(x, x_n))}.\varepsilon \\ &< \frac{\varepsilon}{\phi(\varepsilon)}\varepsilon \end{aligned}$$

Therefore, $d^*(Tx, x) = 1$ or $Tx = x$. So, T has fixed point $x \in X$. Now, we will prove that T has unique fixed point. Assume that T has two distinct fixed point $x, y \in X$ with $x \neq y$. Note that

$$d^*(x,y) = d^*(Tx,Ty) \le \frac{d^*(x,y)}{\phi(d^*(x,y))}$$

or $\phi(d^*(x,y)) \le 1$. Therefore, $d^*(x,y) = 1$ or $x = y$. So, T has unique fixed point. ■

Corollary 10.
Let (X,d^*) be a complete multiplicative metric space and $T : X \times X$ be a mapping. Suppose there exist continuous non-decreasing function $\phi : [1,\infty) \to [1,\infty)$ with $\phi(t) = 1$ if and only if $t = 1$ such that

$$d^*(T^n x, T^n y) \le \frac{d^*(x,y)}{\phi(d^*(x,y))} \qquad (8)$$

for all $x, y \in X$ and $n \in \mathbf{N}$, then T has unique fixed point.

Proof.

Let $S = T^n$. Therefore, by Theorem 9, there exist fixed point of S $z \in X$ such that $Sz = z$. So, $T^n z = z$.
Note that

$$d^*(Tz,z) = d^*(T(T^n z), z) = d^*(T^n(Tz), T^n z) \le \frac{d^*(Tz,z)}{\phi(d^*(Tz,z))}$$

We get $\phi(d^*(Tz,z)) = 1$ or $d^*(Tz,z) = 1$. Hence, $Tz = z$ or T has fixed point.
To prove uniqueness, assume that T has two fixed point $y, z \in X$ with $y \ne z$. We have

$$d^*(y,z) = d^*(T^n y, T^n z) \le \frac{d^*(y,z)}{\phi(d^*(y,z))}$$

Hence, we get $y = z$ which contradict with $y \ne z$. So, T has a unique fixed point. ■

The following example are given to support our result related to weakly multiplicative contraction mapping.

Example 2.
Let $X = [1,2]$ and multiplicative metric $d^* : X \times X \to \mathbf{R}^+$ defined by

$$d^*(x,y) = e^{2|x-y|}$$

for all $x, y \in X$. It is easy to check that (X,d^*) is complete multiplicative metric spaces. Let continuous nondecreasing function $\phi : [1,\infty) \to [1,\infty)$ with $\phi(t) = t^{\frac{1}{4}}$ that satisfied $\phi(t) = 1$ if and only if $t = 1$ is control function. We define $Tx = 2x - \frac{1}{2}x^2$ for all $x \in X$. Note that, without loss our generality, let $x > y$

$$\begin{aligned} d^*(Tx,Ty) &= d^*\left(2x - \tfrac{1}{2}x^2, 2y - \tfrac{1}{2}y^2\right) \\ &= e^{\left|2x - \frac{1}{2}x^2 - \left(2y - \frac{1}{2}y^2\right)\right|} \\ &= e^{2(x-y) - \frac{2}{4}\left(x^2 - y^2\right)} \\ &= \frac{e^{2(x-y)}}{e^{\frac{2}{4}\left(x^2-y^2\right)}} \\ &\le \frac{e^{2(x-y)}}{e^{\frac{2}{4}(x-y)^2}} \\ &\le \frac{e^{2|x-y|}}{e^{\frac{2}{4}|x-y|}} \\ &\le \frac{d^*(x,y)}{\phi(d^*(x,y))} \end{aligned}$$

It shows that T is satisfied condition of weakly multiplicative contraction mapping. According to Theorem 9, then T has a unique fixed point $x = 2 \in X$.

IV. CONCLUSION

In this paper, some fixed point theorems are proved related to multiplicative metric spaces. Also, some examples of that theorem are be given to support main results. Furthermore, our results can be generalized for other contraction mapping for future works.

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