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Mathematical Model of Armed Criminal Group with Pre-emitive and Repressive Intervention

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Abstract. *Armed Criminal group is one of the problems faced by many countries in the world. Awful behaviour of armed criminal group members can affect a large amount of people. In this paper, we construct a deterministic mathematical model that takes into account persuasive and repressive intervention. We consider crime as a social epidemic. We determine the armed criminal group free equilibrium point and the armed criminal group persistence equilibrium point together with their existence condition. The next generation matrix is used to obtain the basic reproduction number. The local stability conditions of equilibrium points are proved using linearization. We show that the armed criminal group free equilibrium point is globally asymptotically stable under certain condition. Numerical simulations are performed to support our deductive study.*

Keywords*:armed criminal group, deterministic model, stability analysis , next generation matrix,*

I. INTRODUCTION

Organised crime has a deletrious impact on many countries around the world [1]. They frequently pose serious problems, particularly in urban areas [2]. Armed group, at its most basic level, is an organized group with a clear structure, membership, and the capacity to use aggressivenes in the occupation of its desire [3]. This definition is very general and includes the state armed forces e.g. the police and the army. The armed groups discussed in this article are armed groups that commit crimes. We call it the armed criminal group. Some examples of armed criminal groups are the Yakuza, Triads, drug cartel, and Mafia. They are also called criminal organization [2].

Recently, many mathematical models have been developed to study criminal activity dynamic [4]–[15]. Gonzalez [11] propose a mathematical model of crime by assuming crime as a social epidemic process. The optimal control problem of model of crime is discussed in [6]. A mathematical model of crime that takes into consideration serious and minor criminal activity is discussed in [15]. Jongo [12] study a mathematical model to investigate how minor criminals turn in to major criminals inside and outside of prisons.

Unlike the model previously described, we are specially interested in studying the dynamics of crime caused by individuals or groups (armed criminal groups) when there is persuasive and repressive intervention. Pre-emitive intervention is carried out through education or rehabilitation. On the other hand, repressive intervention is accomplished through punishments e.g. imprisonment.

II. Results and Discussion

2.1 Model Formulation

We assume that the human population is divided into four disjoint subpopulations. P, C_i, C_g, Q represent susceptible human, criminals who commit a crime individually, criminals who commit a crime in groups, and humans who choose not to be a criminal, respectively. We assume that recruitment rate of human (Λ) is constant. Susceptible humans (P) can become criminals due to interactions with criminals (C_g). The effective contact rate between P and C_g is denoted β. P reduces because of natural death at rate μ. Susceptible humans who have received education about criminal act can supress their desire to commit a crime. The proportion of susceptible humans who receive education and the effectivity of education implementation are denoted ξ and ϕ, respectively. Based on these assumptions, we get

$$\frac{dP}{dt} = \Lambda - \beta PM - \mu P - \phi P. \tag{1}$$

The criminals who commit a crime individually increase because of the new criminals i.e. susceptible humans who are influenced by criminals. After committing several crimes individually, the criminals will decide whether they continue or stop doing crimes. We assume that someone who prefer doing crime will commit a crime in group. On the other hand, someone who prefer to stop doing a crime because of education or other factors go to Q compartment. Sometimes, a criminals are killed due to their crimes.

Therefore, we consider using crime induced death rate μ_x. Based on these assumptions, we obtain

$$\frac{dO}{dt} = \beta PM - (\mu + \mu_x + \gamma + \phi)O. \quad (2)$$

The criminals who commit a crime in group increase because of criminals who usually commit a crime individually join the group. Similar to O compartment, we assume that criminals who commit crimes in group can be killed due to their crimes and can stop doing a crime at rate σ. Hence, we get

$$dM = \gamma\theta O - (\sigma + \mu + \mu_x)M, \quad (3)$$

$$\frac{dQ}{dt} = \gamma(1-\theta)O + \sigma M - \mu Q + \phi(P+O). \quad (4)$$

θ is the proportion of O who go to M compartment.

2.2 Basic Properties

Theorem 1*. Solutions of system* (-) with non-negative initial value are always non-negative.

Proof*.*

Regard as $G(t) = \min\{P(t), O(t), M(t), Q(t)\}$. Its clear that $G(0) \ge 0$. Suppose that there is t^* such that $G(t) \ge 0$ for $t \in [0, t^*)$, $G(t^*) = 0$, and $G(t^+) < 0$ for $t^+ > t^*$. Assume that $G(t) = P(t)$, from equation , we obtain

$$\begin{aligned} \frac{dP(t^*)}{dt} &= \Lambda - \beta P(t^*)M - \mu P(t^*) - \phi P(t^*) \\ &= \Lambda \\ &> 0. \end{aligned} \quad (5)$$

Its clear that $P(t^+) > 0$ for $t^+ > t^*$ which leads to a contradiction. Thus, $P(t)$ is always non-negative for $t \ge 0$. By similar method, we can show that $O(t), M(t), Q(t)$ are non-negative for $t \ge 0$.

Therorem 2. *Solutions of system (-) are bounded*

Proof. Let N is the total number of human. Hence, $N = P + O + M + Q$ and

$$\frac{dN}{dt} = \frac{dP}{dt} + \frac{dO}{dt} + \frac{dM}{dt} + \frac{dQ}{dt}. \quad (6)$$

From the system (-), we get

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \mu N - \mu_x(O+M) \\ &\le \Lambda - \mu N. \end{aligned} \quad (7)$$

Hence, a standard comparison argument provides

$$\lim_{t\to\infty} \sup N(t) \le \frac{\Lambda}{\mu}.$$

This indicates that the solutions of the system are bounded for $t \ge 0$. Based on Theorem 1 and Theorem 2, we obtain feasible region of the system as follows.

$$\Omega = \left\{(P, O, M, Q) \mid 0 \le P + O + M + Q \le \frac{\Lambda}{\mu}, P \ge 0,, O \ge 0, M \ge 0, Q \ge 0\right\}$$

2.3 Equilibrium points and the basic reproduction number

The armed criminal group free equilibrium point is $\Xi_0 = (P^*, O^*, M^*, Q^*) = \left(\frac{\Lambda}{\phi+\mu}, 0, 0, 0\right)$. Ξ_0 is always exists in $^4_+$.

The armed criminal group persistence equilibrium point is $\Xi_1 = (P^{**}, O^{**}, M^{**}, Q^{**})$ where

$$P^{**} = \frac{\gamma\theta\Lambda - (\mu + \mu_x + \gamma + \phi)(\sigma + \mu + \mu_x)M^{**}}{\gamma\theta(\mu+\phi)},$$

$$O^{**} = \frac{(\sigma + \mu + \mu_x)M^{**}}{\gamma\theta},$$

$$M^{**} = \frac{(R_0 - 1)(\mu+\phi)(\mu + \mu_x + \gamma + \phi)(\sigma + \mu + \mu_x)}{\beta(\mu + \mu_x + \gamma + \phi)(\sigma + \mu + \mu_x)},$$

$$Q^{**} = N^* - (P^{**} + O^{**} + M^{**}),$$

$$N^* = \frac{\Lambda}{\mu} - \frac{\mu_x(O^{**} + M^{**})}{\mu}.$$

It is clear that Ξ_1 exists in $^4_+$ if $R_0 > 1$.

The basic reproduction number is determined by using next generation matrix. From , and Ξ_0, we get

$$R_0 = \rho(FV^{-1}) = \frac{\beta\Lambda\gamma\theta}{(\phi+\mu)(\mu + \mu_x + \phi + \gamma)(\sigma + \mu + \mu_x)}.$$

2.3 Stability of the armed criminal group free equilibrium points

Theorem 3. *Armed criminal group free equilibrium point Ξ_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. We will prove this theorem by using linearization method. The Jacobian matrix of the system (-) is

$$J = \begin{pmatrix} -\beta M - (\mu+\phi) & 0 & -\beta P & 0 \\ \beta M & -(\mu + \mu_x + \phi + \gamma) & \beta P & 0 \\ 0 & \gamma\theta & -(\sigma + \mu + \mu_x) & 0 \\ \phi & \gamma(1-\theta) + \phi & \sigma & -\mu \end{pmatrix}. \quad (8)$$

Subtituting Ξ_0 into J, we obtain

$$J(\Xi_0) = \begin{pmatrix} -(\mu+\phi) & 0 & -\beta P^* & 0 \\ 0 & -(\mu + \mu_x + \phi + \gamma) & \beta P^* & 0 \\ 0 & \gamma\theta & -(\sigma + \mu + \mu_x) & 0 \\ \phi & \gamma(1-\theta) + \phi & \sigma & -\mu \end{pmatrix},$$

where $P^* = \frac{\Lambda}{\mu+\phi}$. The eigen values of $J(\Xi_0)$ are the roots of characteristics polynomial

$$\begin{aligned} H(\lambda) &= \det(J(\Xi_0) - \lambda I_{4\times4}) \\ &= (-\mu - \lambda)(-(\mu+\phi) - \lambda)H_1(\lambda), \end{aligned}$$

where

$$\begin{aligned} H_1(\lambda) &= (-(\mu + \mu_x + \phi + \gamma) - \lambda)(-(\sigma + \mu + \mu_x) - \lambda) - \beta P^* \gamma\theta, \\ &= \lambda^2 + ((\mu + \mu_x + \phi + \gamma) + (\sigma + \mu + \mu_x))\lambda + \left[(\mu + \mu_x + \phi + \gamma)(\sigma + \mu + \mu_x) - \beta P^* \gamma\theta\right]. \end{aligned}$$

It is easy to see that $H(\lambda)$ has two negative roots i.e.

$\lambda_1 = -\mu$ and $\lambda_2 = -(\mu+\phi)$. The other eigenvalues are roots of $H_1(\lambda)$. Based on Descartes' sign rule, $H_1(\lambda)$ has no positive roots if $(\mu+\mu_x+\phi+\gamma)(\sigma+\mu+\mu_x)-\beta P^*\gamma\theta>0$. It is clear that this condition is met if $R_0<1$. Moreover, one eigenvalue is positive provided by $(\mu+\mu_x+\phi+\gamma)(\sigma+\mu+\mu_x)-\beta P^*\gamma\theta<0$. Hence, if $R_0<1$, all eigenvalues of $J(\Xi_0)$ are negative (Ξ_0 is locally asymptotically stable). Furthermore, if $R_0>1$, one eigenvalue of $J(\Xi_0)$ is positive (Ξ_0 is unstable).

Theorem 4. *. Armed criminal group free equilibrium point Ξ_0 is globally asymptotically stable in Ω_+ if $R_0<1$ where*

$$\Omega_+ = \left\{(P,O,M,Q) \mid 0<P+O+M+Q\le \frac{\Lambda}{\mu}, P>0,, O\ge 0, M\ge 0, Q\ge 0\right\}$$

Proof. We will prove this theorem by using Lyapunov direct method. Define $V:\Omega_+ \to$ by

$$V((P,O,M,Q)) = \left(P-P^*-P^*\ln\left(\frac{P}{P^*}\right)\right)+O+\frac{\beta P^*}{\sigma+\mu+\mu_x}M,$$

It is easy to see that $V(\Xi_0)=0$ and $V((P,O,M,Q))>0$ for $(P,O,M,Q)\in\Omega-\{\Xi_0\}$. The derivative of V is

$$\begin{aligned}\frac{dV}{dt} &= \left(1-\frac{P^*}{P}\right)\frac{dP}{dt}+\frac{dO}{dt}+\frac{\beta P^*}{\sigma+\mu+\mu_x}\frac{dM}{dt}\\ &= \frac{1}{P}(P-P^*)(\Lambda-\beta PM-\mu P-\phi P)\\ &\quad +(\beta PM-(\mu+\mu_x+\gamma+\phi)O)\\ &\quad +\frac{\beta P^*}{\sigma+\mu+\mu_x}(\gamma\theta O-(\sigma+\mu+\mu_x)M).\end{aligned}$$

Using the fact that $(\mu+\phi)P^*=\Lambda$, we obtain

$$\begin{aligned}\frac{dV}{dt} &= -\frac{(\mu+\phi)}{P}(P-P^*)^2-\frac{1}{P}(P-P^*)(\beta PM)\\ &\quad +(\beta PM-(\mu+\mu_x+\gamma+\phi)O)\\ &\quad +\frac{\beta P^*}{\sigma+\mu+\mu_x}(\gamma\theta O-(\sigma+\mu+\mu_x)M)\\ &= -\frac{(\mu+\phi)}{P}(P-P^*)^2+(\beta PM-\beta PM)\\ &\quad +\left(\frac{\beta P^*\gamma\theta}{\sigma+\mu+\mu_x}-(\mu+\mu_x+\gamma+\phi)\right)O\\ &\quad +\left(\beta P^*-\frac{\beta P^*}{\sigma+\mu+\mu_x}(\sigma+\mu+\mu_x)\right)M\\ &= -\frac{(\mu+\phi)}{P}(P-P^*)^2+(R_0-1)(\mu+\mu_x+\gamma+\phi)O.\end{aligned}$$

Thus, we obtain $\frac{dV}{dt}\le 0$ if $R_0<1$. Furthermore, $\frac{dV}{dt}=0$ if and only if $P=P^*=\frac{\Lambda}{\phi+\mu}, O=0, M=0, Q=0$. Hence, the largest invariant set contained in $\left\{(P,O,M,Q)\mid \frac{dV}{dt}=0\right\}$ is a singleton set, that is, Ξ_0. Then by Lyapunov-Lasalle theorem, it is proved that Ξ_0 is globally asymptotically stable provided by $R_0<1$.

2.4 stability of armed criminal group persistence equilibrium point

Theorem 5. *The armed criminal group persistence equilibrium point $\Xi_1=(P^{**},O^{**},M^{**},Q^{**})$ is locally asymptotically stable if $R_0>1$ and $\frac{k_1k_2-k_3}{k_1}>0$* where

$k_1 = J_1J_4J_6$,
$k_2 = J_1J_4+J_1J_6+J_4J_6-J_2J_5$,
$k_3 = J_1J_4J_6+J_2J_3J_5-J_1J_2J_5$.
and
$J_1=\beta M^{**}+(\mu+\phi)$,
$J_2=\beta P^{**}$,
$J_3=\beta M^{**}$,
$J_4=(\mu+\mu_x+\phi+\gamma)$,
$J_5=\gamma\theta$,
$J_6=(\sigma+\mu+\mu_x)$,
$J_7=\phi$,
$J_8=\gamma(1-\theta)+\phi$,
$J_9=\sigma$.

Proof. We will prove this theorem by using linearization approach. Subtituting Ξ_1 into , we obtain

$$\begin{aligned}J(\Xi_1) &= \begin{pmatrix}-\beta M^{**}-(\mu+\phi) & 0 & -\beta P^{**} & 0\\ \beta M^{**} & -(\mu+\mu_x+\phi+\gamma) & \beta P^{**} & 0\\ 0 & \gamma\theta & -(\sigma+\mu+\mu_x) & 0\\ \phi & \gamma(1-\theta)+\phi & \sigma & -\mu\end{pmatrix}\\ &= \begin{pmatrix}-J_1 & 0 & -J_2 & 0\\ J_3 & -J_4 & J_2 & 0\\ 0 & J_5 & -J_6 & 0\\ J_7 & J_8 & J_9 & -\mu\end{pmatrix}.\end{aligned}$$

It is clear that $J_i>0$ for $i=1...9$. The eigenvalues of $J(\Xi_1)$ are roots of characteristics polynomial

$$\begin{aligned}K(\lambda) &= \det(\lambda I_{4\times4}-J(\Xi_1))\\ &= (\lambda+\mu)((\lambda+J_1)[(\lambda+J_4)(\lambda+J_6)-J_2J_5]+J_2J_3J_5)\\ &= (\lambda+\mu)([(\lambda+J_1)(\lambda+J_4)(\lambda+J_6)]-(\lambda+J_1)J_2J_5+J_2J_3J_5)\\ &= (\lambda+\mu)K_1(\lambda),\end{aligned}$$

where

$$K_1(\lambda)=\lambda^3+k_1\lambda^2+k_2\lambda+k_3$$

It is convenient to see that $K(\lambda)$ has one negative eigenvalue i.e. $\lambda_1=-\mu$. The other eigenvalues are roots of $K_1(\lambda)$. By using Routh Hurwitz condition, Ξ_1 is stable

if
$$
\begin{aligned}
&k_1 > 0,\\
&\frac{k_1k_2 - k_3}{k_1} > 0,\\
&k_3 > 0.
\end{aligned}
$$

$k_1, k_1k_2 - k_3$ are always positive. It is straightforward to show that $k_3 = (R_0 - 1)(\mu + \phi)(\mu + \mu_x + \gamma + \phi)(\sigma + \mu + \mu_x)$. If $R_0 > 1$ then $k_3 > 0$. Hence, Ξ_1 locally asymptotically stable if $R_0 > 1$.

2.5 Numerical simulations

We now present numerical simulations to support our theoretical results. We performed two scheme numerical simulations to investigate the impact of pre-emitive and repressive intervention. To study the impact of pre-emitive, we conduct numerical simulation with varying ϕ. To investigate the impact of repressive intervention, we perform numerical simulation with varying μ_x. We choose the following set parameter values for illustrative purpose only. These parameter values are selected to show the impact of interventions and to verify our theoretical results.

Parameter	Parameter values
Λ	10
β	0.5
μ	0.3
γ	0.5
θ	0.6
σ	0.4
ϕ	[0,1)
μ_x	[0,1)

2.5.1. Dynamics of armed criminal group without interventions

In this part, we performed numerical simulation using parameter value that are mentioned in **Table 1** with $\phi = 0$ and $\mu_x = 0$. The reproduction number $R_0 = 8.9286 > 1$. Based on **Theorem 3**, the armed criminal goups persist.

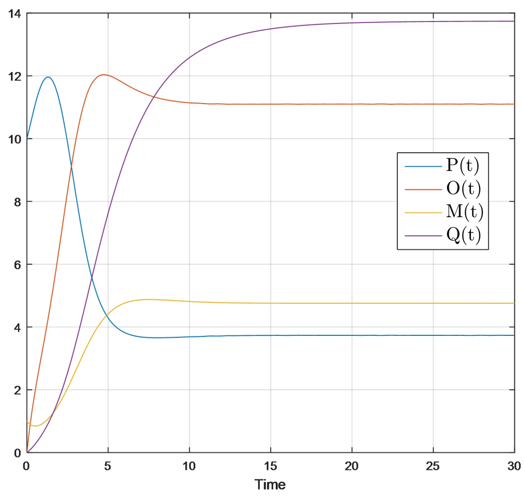

Figure 1. Dynamics of armed criminal group for $R_0 = 8.9286 > 1$

Figure 1 show the dynamics of armed criminal groups. It is clear that the criminals who commit crimes individually or in group always persist in high density all the time eventhough some criminals quit and never commit a crime again. This condition is certainly worrisome.

2.5.2. Impact of pre-emitive intervention only

In this part, we performed numerical simulation with varying ϕ and $\mu_x = 0$. We use $\phi = 0.2, 0.6,$ and 0.95. The reproduction numbers are 4.2857,1.7007, and 0.9796. Based on **Theorem 5**, the criminal compartments i.e O and M will converge to armed criminal group persistence equilibrium for the first and second case. On the other hand, the criminal compartments converge to armed criminal group free equilibrium point for the third case.

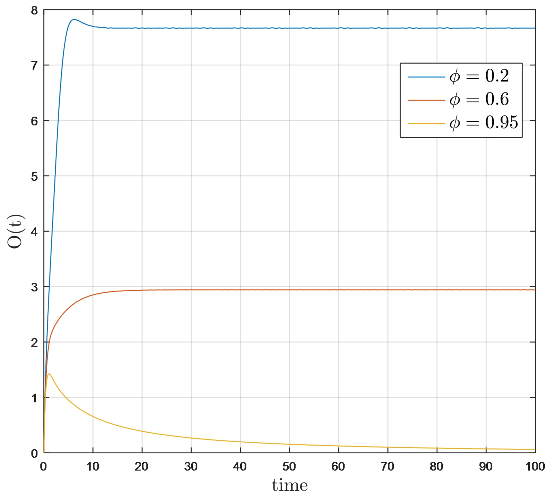

Figure 2. Dynamics of O compartment for varying ϕ

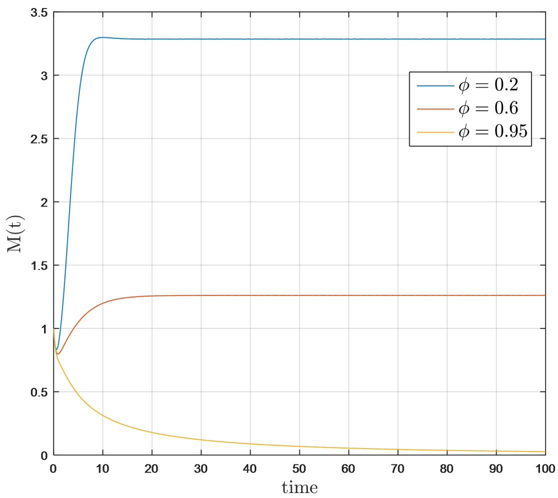

Figure 3. Dynamics of M compartment for varying ϕ

Figure 2-3 show that the criminals who commit crimes individually or in groups decrease as ϕ increases. The blue and red curve converge to a positive equilibrium. On the other hand, the yellow curve converge to zero. These results indicate that the criminals persist for the first and second case and extinct for the third case.

2.53. Impact of repressive intervention only

In this part, we performed numerical experiment with varying μ_x and $\phi = 0$. We use $\mu_x = 0.2, 0.6,$ and 0.95. The reproduction numbers are $5.5556, 2.7473,$ and 1.7316. Based on **Theorem 5**, the criminal compartments i.e O and M will converge to armed criminal group persistence equilibrium for all cases.

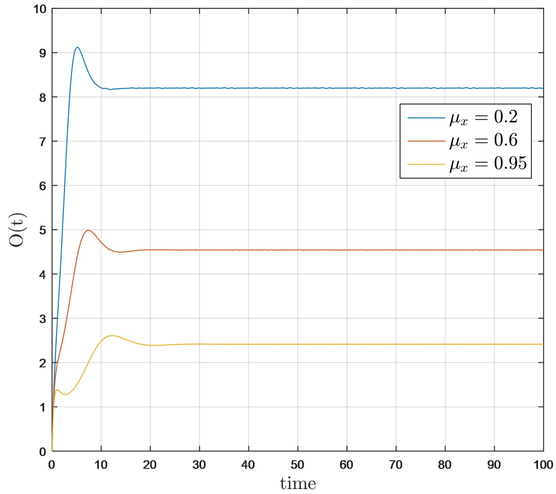

Figure 4. Dynamics of O compartment for varying μ_x

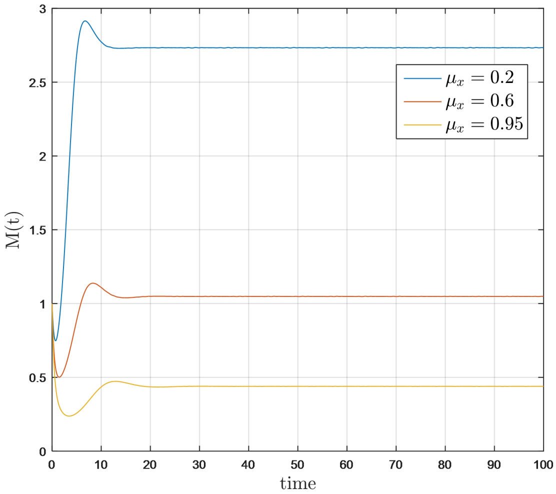

Figure 5. Dynamics of M compartment for varying μ_x

Figure 4-5 show that the criminals who commit crimes individually or in groups decrease as μ_x increases. Unlike the previous simulation, these results indicate that the criminals persist for all cases.

2.53. Impact of pre-emitive and repressive intervention

In this part, we performed numerical experiment with varying μ_x and ϕ. We use $\mu_x = 0.1, 0.25, 0.5,$ and $\mu_x = 0.1, 0.25, 0.5$. The reproduction numbers are $4.6875, 2.2083,$ and 0.8681. Based on **Theorem 5**, the criminal compartments i.e O and M will converge to armed criminal group persistence equilibrium for the first and secod case.

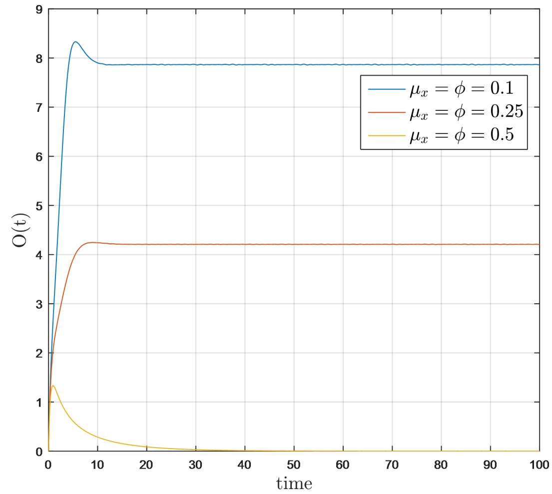

Figure 6. Dynamics of O for varying μ_x and ϕ

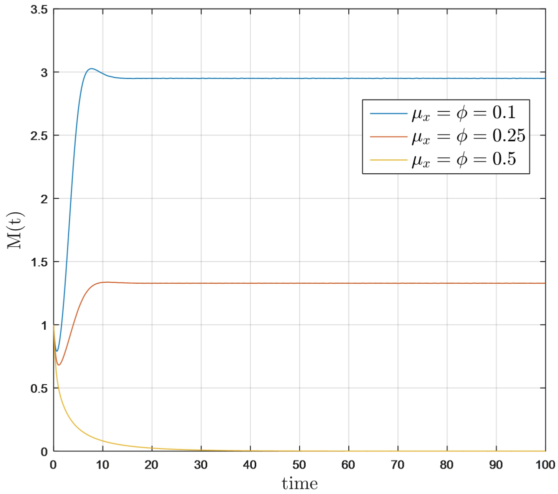

Figure 7. Dynamics of M for varying μ_x and ϕ

Figure 6-7 show that the criminals who commit crimes individually or in groups decrease as ϕ increases. The blue and red curve converge to a positive equilibrium. On the other hand, the yellow curve converge to zero. These results indicate that the criminals persist for the first and second case and extinct for the third case.

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