

Stability Analysis of Tuberculosis SITS Model

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Abstract. Tuberculosis (TB) is an infectious disease caused by mycobacterium tuberculosis. The purpose of this study is to investigate the dynamics of TB spread by using mathematical model. We develop SITS model which expressed as system of differential equations. The system has two equilibrium points, namely disease-free equilibrium point and endemic equilibrium point. The stability condition of the equilibrium points is proved. We perform several numerical simulations to support our theoretical results.

Keywords: tuberculosis, mathematical model, stability analysis

I. INTRODUCTION

Tuberculosis (TB) is an infectious disease caused by Mycobacterium tuberculosis. TB is one of the deadliest diseases and transmitted through droplets (sputum splashes) [1].

TB is the most dangerous infectious disease in the world. The World Health Organization (WHO) reported that 1.5 million people died from TB (1.1 million is HIV negative and 0.4 million is HIV positive), with 89,000 men, 480,000 women and 140,000 children. It is estimated that 9.6 million people suffer from TB in 2014 with 12% of whom are HIV-positive [2]. WHO stated that 58% of new TB cases occurs in Southeast Asia and Western Pacific region in 2014. The number of TB cases in India, Indonesia and China are 23%, 10% and 10% of the total incidence worldwide, respectively. It is estimated that one million new cases per year occur in Indonesia (WHO, 2015). In West Sulawesi, the number of TB patients in 2007 is 5,100 clinical TB patients. The number of TB clinical patients in 2007 had increased compared to 2006. One intervention used by the West Sulawesi Provincial government in controlling TB is increasing the effectiveness of TB control through strengthening DOTS [3]. In recent years, a deterministic model is used study tuberculosis spread e.g. SEI model [4], SIR and SEIR models [5] and SITS model [6].

In this paper, we discuss SITS (Susceptible-Infected-Treatment-Susceptible) model. We divide the population into three subpopulations, i.e. susceptible individuals, infected individuals, and individuals who take medication. (Treatment). We assume that human who recover from TB can get reinfected.

II. Model Formulation and Equilibrium Points

2.1 Model Formulation

In this study, SITS model incorporating the effect of vaccination is formulated. We determine the equilibrium

points of the system and stability condition of the equilibrium point. The assumptions made to formulate the model are:

- Recruitment of susceptible individuals is constant.
- Susceptible individual who interacts with infected individual can be infected.
- Infected individual can die due to TB.
- Human who recover from TB can get reinfected.
- The population is homogen, which implies the death rate and transmission rate of all individuals are equal.

The parameter used are presented in Table 1.

Table 1. Description of Model Parameter

Symbol	Description
ρ	vaccination coverage
α	recruitment rate
β	transmission rate
μ_1	natural death rate
μ_2	disease induced death rate
γ	treatment rate
ω	recovery rate

Based on the assumptions that have been made, we obtain a diagram as follows

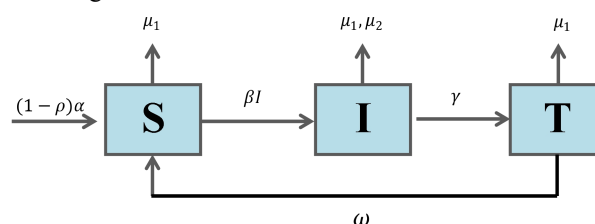


Figure 1. Diagram of SITS Model

Based on Figure 1, we obtain the following system of differential equations

$$\begin{aligned} \frac{dS}{dt} &= (1-\rho)\alpha + \omega T - (\beta I + \mu_1)S, \\ \frac{dI}{dt} &= \beta IS - (\gamma + \mu_1 + \mu_2)I, \\ \frac{dT}{dt} &= \gamma I - (\mu_1 + \omega)T. \end{aligned} \quad (1)$$

2.2 Equilibrium Points

The system has two equilibrium points. The disease free equilibrium point and endemic equilibrium point are respectively

$$x_{dfe}(S^*, I^*, T^*) = (S^*, 0, 0) = \left(\frac{(1-\rho)\alpha}{\mu_1}, 0, 0 \right), \quad (2)$$

and

$$x_{ee} = (S^{**}, I^{**}, T^{**}),$$

where

$$\begin{aligned} S^{**} &= \frac{(1-\rho)\alpha(A(\mu_1 + \omega)) - \omega\gamma D}{(\mu_1 + \omega)(-\beta D + \mu_1 A)}, \\ I^{**} &= -\frac{D}{A}, \\ T^{**} &= -\frac{\gamma D}{A(\mu_1 + \omega)}. \\ A &= \beta\omega \left(\frac{\gamma}{(\mu_1 + \omega)} \right) - \beta(\gamma + \mu_1 + \mu_2), \\ D &= \beta\alpha(1-\rho) - \mu_1(\gamma + \mu_1 + \mu_2). \end{aligned}$$

III. Model Analysis

We use next generation approach [7] to determine the basic reproduction number. The reproduction number of the model is

$$R_0 = \frac{\beta(1-\rho)\alpha}{\mu_1(\gamma + \mu_1 + \mu_2)}.$$

Now, we present stability condition of the disease free equilibrium point.

Teorema 2. x_{dfe} is locally asymptotically stable If $R_0 < 1$.

x_{dfe} is unstable if $R_0 > 1$.

Proof. The Jacobian matrix of the model at x_{dfe} is

$$J_{dfe} = \begin{bmatrix} -\mu_1 & -\beta \left(\frac{(1-\rho)\alpha}{\mu_1} \right) & \omega \\ 0 & \beta \left(\frac{(1-\rho)\alpha}{\mu_1} \right) - (\gamma + \mu_1 + \mu_2) & 0 \\ 0 & \gamma & -(\mu_1 + \omega) \end{bmatrix}.$$

The characteristic polynomial of J_{dfe} is

$$(-\mu_1 - \lambda) \left((\gamma + \mu_1 + \mu_2)(R_0 - 1) - \lambda \right) (-(\mu_1 + \omega) - \lambda) = 0.$$

It is clear that the eigenvalues of J_{dfe} are

$$\begin{aligned} \lambda_1 &= -\mu_1, \\ \lambda_2 &= -(\mu_1 + \omega), \\ \lambda_3 &= (\gamma + \mu_1 + \mu_2)(R_0 - 1). \end{aligned} \quad (3)$$

It easy to see that all eigenvalues are negative if $R_0 < 1$.

Hence, The disease free equilibrium point x_{dfe} is locally asymptotically stable and unstable if $R_0 < 1$ and $R_0 > 1$, respectively.

Now, we analyze stability condition of x_{ee} . The Jacobian matrix of the model at x_{ee} is

$$J_{ee} = \begin{bmatrix} \frac{\beta D}{A} - \mu_1 & -\beta \left(\frac{(1-\rho)\alpha(A(\mu_1 + \omega)) - \omega\gamma D}{(\mu_1 + \omega)(-\beta D + \mu_1 A)} \right) & \omega \\ \frac{\beta D}{A} & \beta \left(\frac{(1-\rho)\alpha(A(\mu_1 + \omega)) - \omega\gamma D}{(\mu_1 + \omega)(-\beta D + \mu_1 A)} \right) - (\gamma + \mu_1 + \mu_2) & 0 \\ 0 & \gamma & -(\mu_1 + \omega) \end{bmatrix}.$$

We set

$$\begin{aligned} E_0 &= \frac{\beta D}{A} - \mu_1, \\ F_0 &= \beta \left(\frac{(1-\rho)\alpha(A(\mu_1 + \omega)) - \omega\gamma D}{(\mu_1 + \omega)(-\beta D + \mu_1 A)} \right), \\ J_0 &= (\mu_1 + \omega), \\ G_0 &= \frac{\beta D}{A}, \\ H_0 &= \beta \left(\frac{(1-\rho)\alpha(A(\mu_1 + \omega)) - \omega\gamma D}{(\mu_1 + \omega)(-\beta D + \mu_1 A)} \right) - (\gamma + \mu_1 + \mu_2), \end{aligned}$$

which implies

$$J_{ee} = \begin{bmatrix} E_0 & -F_0 & \omega \\ -G_0 & H_0 & 0 \\ 0 & \gamma & -J_0 \end{bmatrix}.$$

The characteristic polynomial of J_{ee} is

$$\lambda^3 - (E_0 + H_0 - J_0)\lambda^2 + (E_0H_0 - E_0J_0 - H_0J_0 - F_0G_0)\lambda + (E_0H_0J_0 + F_0G_0J_0 + \omega\gamma G_0) = 0.$$

Based on Routh Hurwitz condition [8], all eigenvalues have negative real part if $K_0 < 0$, $M_0 > 0$, and

$$\frac{(K_0L_0) + M_0}{K_0} < 0,$$

where

$$K_0 = (E_0 + H_0 - J_0),$$

$$L_0 = (E_0H_0 - E_0J_0 - H_0J_0 - F_0G_0),$$

$$M_0 = (E_0H_0J_0 + F_0G_0J_0 + \omega\gamma G_0).$$

Hence, we get the following theorem.

Theorem 3. x_{ee} is locally asymptotically stable if $K_0 < 0$,

$M_0 > 0$, and $\frac{(K_0L_0) + M_0}{K_0} < 0$.

IV. Numerical Simulations

To verify and to support our theoretical results, we perform numerical simulation using the following parameter value

Symbol	Parameter values	
	Simulation I $R_0 = 0.4527 < 1$	Simulation II $R_0 = 4.527 > 1$
ρ	0.45	0.45
α	0.45	0.45
β	0.075	0.75
μ_1	0.1	0.1
μ_2	0.01	0.01
γ	0.30	0.3
ω	0.1	0.1

The initial value used is $S(0) = 10, I(0) = 1, T(0) = 0$. The result is shown in Figure 2 and Figure 3. It is clear that the infected and treatment compartment tend to zero and positive equilibrium if $R_0 < 1$ and $R_0 > 1$, respectively.

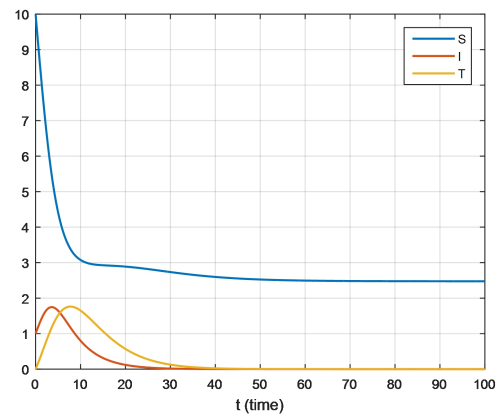


Figure 2. The solution curves with $R_0 = 0.4527 < 1$.

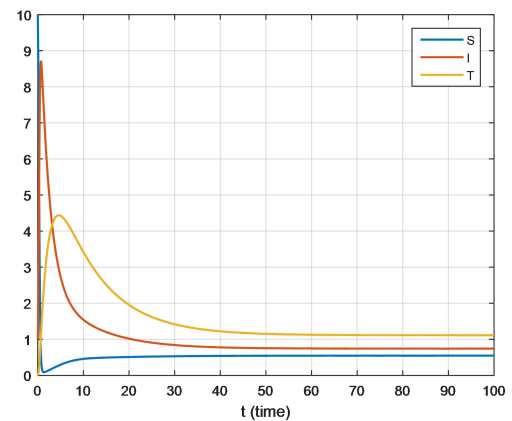


Figure 3. The solution curves with $R_0 = 4.527 > 1$.

Now, we present numerical simulation result with parameter values presented in Table 2 and varying γ . The result is shown in the following figure.

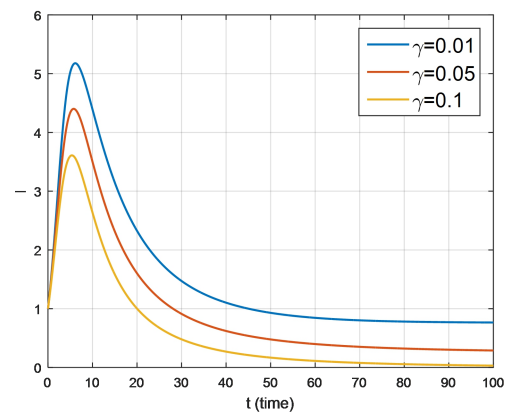


Figure 4. The solution curves with varying γ

Figure 4 shows that TBC prevalence can be reduced by increasing recovery rate (γ). This result shows that the government should give health education related TBC because this intervention can increase public awareness on the risks of TBC which can increase early detection of TBC.

V. Conclusion

In this paper, tuberculosis SITS model is discussed. We prove stability condition of all equilibrium points. Intervention which can increase treatment rate e.g. health education related TB can reduce TB prevalence.

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